# Adaptive Control for Regulation of a Quadratic Function of the State

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***Abstract*—We propose an adaptive controller for a dynamical system, where we need a quadratic function of the state to track a given reference signal. This problem appears in the control of generators in weak-grid conditions for example. While the controller can measure the tracking error, the main difficulty arises from the fact that the parameters of the quadratic function itself are not known to the controller. Our approach consists of simultaneously estimating the quadratic function while tracking the reference signal, similar to the approach employed in adaptive control. The quadratic structure of the tracking function necessitates, however, a new adaptive law for estimating the parameters. Even though estimation and control are in general two contradicting requirements, using this new adaptive law and a multilevel controller we prove that the tracking error converges to zero in the absence of measurement noise. In the presence of bounded noise we show that the tracking error can be driven to a neighborhood around the origin.**

## I. Introduction

The focus of this paper is a control problem where a quadratic function of the state with unknown parameters is required to track a reference signal. The quadratic feature of the tracking requirement causes standard linear controllers to be inadequate since the direction in which the state should advance to in order to reduce the tracking error is a nonlinear function of that error. This can limit the tracking objective from being achieved globally. The presence of the unknown parameters further exacerbates the problem, since it prohibits the desired state-value from being computed off-line. Also these parametric uncertainties may alter the underlying plant dynamics significantly so that a sequential action of system identification followed by control may not be sufficient. What may be called for is therefore an adaptive approach that simultaneously takes identification and control actions.

Nonlinear adaptive control has been investigated extensively during the past few decades, see [1]–[7] for example. However, these methods, which focus on unknown nonlinearities in the dynamics, rather than in the tracking function, prove to be inadequate to the problem structure we consider. The approach that is proposed in this paper is a significant departure from the above methods and is able to accommodate the constraints of the problem at hand and solve the tracking problem. In particular, the proposed approach consists of three components, the combined execution of all of which is the main contribution of this paper. The first component is the estimation of the unknown parameters. Standard parameter identification techniques [8] are inapplicable since the loss of excitation as the system converges, in combination with measurement noise, may cause their estimates to diverge. Motivated by techniques from adaptive control [9], [10], we use a parameter estimation technique that guarantees convergence to some values with which the tracking function is able to follow the reference signal.

The second component of our approach is to use the estimate of the unknown parameters, while it is constantly being updated, to compute the desired steady-state solution that will result in zero tracking error. Our approach guarantees that the estimated parameters will always converge to values for which the underlying quadratic equation, which may not have a solution everywhere, does have a solution. The third component is to use an already developed controller to drive the state of the plant to the desired steady-state solution computed by the second component.

In §II we state a general control problem where a nonlinear function of the state is to track a reference signal. We then present in §III our approach for solving this problem, and state the main theorem. In §IV we prove the main theorem and the properties of our approach. In §V we show the applicability of our approach in a practical setting.

## II. Problem Statement

### A. Notations

We use $j$ for $\sqrt{-1}$, $|\cdot|$ for the absolute value of real or complex scalars, $\bar{\cdot}$ for the conjugate operation of a complex number, and $\mathrm{Re}\,\cdot$ and $\mathrm{Im}\,\cdot$ for the real and imaginary parts of a complex number, respectively. We use $\|\cdot\|$ for the 2-norm of a real vector and for the induced 2-norm of a matrix, and $\|\cdot\|_F$ for the Frobenius norm of a matrix. The identity matrix is $I$. A function $\gamma : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$. We define $B_\varepsilon(\mathcal{P}) \doteq \{x \,|\exists y \in \mathcal{P}, \|x-y\| \le \varepsilon\}$ where $\mathcal{P}$ is a set and $\varepsilon$ is a nonnegative number, and define $\sigma_{\min,>0}(M)$ to be the smallest nonzero singular value of a matrix $M$.

### B. Plant Model

We consider a continuous-time control system, $t \in \mathbb{R}_{\ge 0}$:

$$\begin{aligned}\dot{x}(t) &= f_x(x(t), \xi(t), u(t)) \\ \dot{\xi}(t) &= f_\xi(x(t), \xi(t), u(t))\end{aligned} \tag{1}$$

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$$y(t) = H_0 x(t) + d_0 \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $\xi(t) \in \mathbb{R}^{n'}$ are the state of the system, $u(t) \in \mathbb{R}^p$ the control input, and $y(t) \in \mathbb{R}^m$ a dependent signal. The values of $H_0$ and $d_0$ are unknown, but the set $\mathcal{P}_0 \subseteq \mathbb{R}^{m\times n} \times \mathbb{R}^m$ to which they belong is known to the controller. The functions $f_x$ and $f_\xi$ may also be unknown.

The control objective is to have $c(x(t), y(t))$, where $c(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a known continuous function, track an external reference signal $r(t) \in \mathcal{R} \subset \mathbb{R}^n$. For the control objective to be attainable, the set $\mathcal{R}$ must be defined such that for every $r \in \mathcal{R}$, and for every $(H_0, d_0) \in \mathcal{P}_0$, there exists $x(t, H_0, d_0)$ such that $c(x, H_0 x + d_0) = r$.

The signals available to the controller are noisy measurements of $x$ and $y$:

$$\underline{x}(t) = x(t) + v(t), \qquad \underline{y}(t) = y(t) + w(t)$$

where $v$ and $w$ are unknown measurement noise. We do, however, assume the bounds $v_{\max}$ and $w_{\max}$ such that

$$\|v(t)\| \le v_{\max}, \quad \|w(t)\| \le w_{\max}, \qquad \forall t$$

are known by the controller.

*Remark*: A solution to this problem, where $c$ is a general nonlinear function, is provided in §III and proved in §IV. In §V we limit our attention of $c$ to a bilinear function of $x$ and $y$, making $c$ a quadratic function of the state.

## III. Proposed Controller

### A. Definitions and Assumptions

With our controller we will be reducing this problem to that of driving the state $x$ to some reference signal $\check{x}$, conforming to a structure which is common in the adaptive control literature. As the focus of this paper is addressing the unknown nonlinear tracking function, we assume that a solution already exists for the reduced problem:

*Assumption 1:* There exists a controller, $\mathcal{C}_1$, whose input signals are $\underline{x}$ and a reference signal $\check{x}$, and its output signal is $u$. When the plant (1) is used with this controller $x$ and $\xi$ are bounded for every bounded reference signal $\check{x}$, and there exists a function of class $\mathcal{K}$, $\gamma(\cdot)$, such that if $\lim_{t\to\infty} \check{x}(t)$ exists then

$$\limsup_{t\to\infty} \|x(t) - \check{x}(t)\| \le \gamma\big(\limsup_{t\to\infty} \|v(t)\|\big). \tag{3}$$

Note that any controller that renders the closed-loop system input-to-state stable (ISS) satisfies this assumption [11, §3.1]. In particular, for a linear time-variant system whose dynamics may be uncertain but are known to belong to a closed set, if a linear controller with a common quadratic Lyapunov function over the whole set can be found, as in [12, Ch. 7], that controller will render the closed-system to be ISS [11, §3.3]. In this case the gain $\gamma$ in (3) will be linear.

The definitions below are used to describe our solution

a) Define the function $g(H, d, r)$ such that for every $H$, $d$ and $r$ for which the control objective can be attained, i.e. there exists $x$ such that $c(x, Hx + d) = r$, $g$ satisfies

$$c(g(H, d, r), Hg(H, d, r) + d) = r. \tag{4}$$

In other words $g$ returns an $x$, which may not necessarily be unique, that satisfies the control objective.

b) Define the set $\mathcal{P}_2(r)$, $r \in \mathcal{R}$ as a domain of $H$ and $d$ on which $g(\cdot,\cdot,r)$ is defined, and such that $\mathcal{P}_0 \subset \mathcal{P}_2(r)$. This means that (4) holds for all $(H, d) \in \mathcal{P}_2(r)$. Note that by the definition of $\mathcal{R}$ such a set exists.

c) Define the set $\mathcal{P}_3$ as the convex hull of $\mathcal{P}_0$.

d) For any $x_s \in \mathbb{R}^n$ define

$$X_s \doteq \left[\begin{array}{ccc|c} x_s^T & 0 & 0 & \\ 0 & \ddots & 0 & I \\ 0 & 0 & x_s^T & \end{array}\right]. \tag{5}$$

We then make a technical assumption to ensure the regularity of the objects just defined.

*Assumption 2:*

a) The function $g$ is continuous over $\mathcal{P}_2(r)$ $\forall r \in \mathcal{R}$.

b) The set $\mathcal{P}_0$, and therefore the set $\mathcal{P}_3$, is closed and bounded.

c) The set $\mathcal{P}_2(r)$ is closed for all $r \in \mathcal{R}$ and $\mathcal{P}_2(r)$ is a continuous set-valued map over $\mathcal{R}$.

d) The convex set $\mathcal{P}_3$ can be described by a set of $m$ differentiable convex functions, $f_1, \dots f_m$, $f_i : \mathbb{R}^{m(n+1)} \to \mathbb{R}$, and an affine function, $h : \mathbb{R}^{m(n+1)} \to \mathbb{R}^q$, such that $(H, d) \in \mathcal{P}_3$ if and only if

$$h(H, d) = 0 \text{ and } f_i(H, d) \le 0, \ \forall i = 1, \dots, m.$$

Furthermore, Slater's constraint qualification [13, §5.2.3] holds: $\exists\ (H_s, d_s) \in \mathcal{P}_3$ such that $f_i(H_s, d_s) < 0$, $\forall i = 1, \dots, m$.

When estimating the unknown parameters, in order to show convergence the search space, $\mathcal{P}_3$, must be convex. The following additional assumption is only needed when a convex set $\mathcal{P}_3$ such that $\mathcal{P}_3 \subset \mathcal{P}_2(r)$, $\forall r \in \mathcal{R}$, does not exist.

*Assumption 3:*

a) There exists $x_\nu \in \mathbb{R}^n$ such that if $(H_0, d_0) \in \mathcal{P}_0$ and $(H, d) \in \mathcal{P}_3$ satisfy

$$Hx_\nu + d = H_0 x_\nu + d_0, \tag{6}$$

then necessarily $(H, d) \in \mathcal{P}_0$.

b) There exists $\varepsilon > 0$ such that $B_{\epsilon_\nu + \varepsilon}(\mathcal{P}_0) \cap \mathcal{P}_3 \subset \mathcal{P}_2(r)$ $\forall r \in \mathcal{R}$ where

$$\epsilon_\nu \doteq \frac{1}{\sigma_{\min,>0}(X_\nu)} \Big(\delta(w_{\max}, v_{\max}) + 2 \sup_{(H,d)\in\mathcal{P}_3} \|H\| \big(\gamma(v_{\max}) + v_{\max}\big)\Big)$$

and

$$\delta(w_{\max}, v_{\max}) \doteq 2\big(w_{\max} + v_{\max} \sup_{(H,d)\in\mathcal{P}_0} \|H\|\big). \tag{7}$$

This assumption also implies the existence of a set $\mathcal{P}_1$ such that $B_{\epsilon_\nu + \varepsilon'}(\mathcal{P}_0) \cap \mathcal{P}_3 \subset \mathcal{P}_1$ and $B_{\varepsilon''}(\mathcal{P}_1) \cap \mathcal{P}_3 \subset \mathcal{P}_2(r)$ $\forall r$ for some $\varepsilon' > 0$, $\varepsilon'' > 0$, $\varepsilon' + \varepsilon'' < \varepsilon$.

In the sequel, we will simply write $\delta$ as a shorthand for $\delta(w_{\max}, v_{\max})$. Note that in the absence of measurement noise $\epsilon_\nu = \delta = 0$. In this case Assumption 3b simply requires that $\mathcal{P}_0$, which by the definition of $\mathcal{R}$ is contained in $\mathcal{P}_2(r)$ for all $r \in \mathcal{R}$, is contained in *the interior of* $\mathcal{P}_2(r)$, where this interior is defined in relative to $\mathcal{P}_3$.

Assumptions 1-3 characterize the most general family of problems to which the proposed controller is applicable. In §V we present a practical example and show that all these assumptions hold.

### B. Solution

We use a two-level control approach to solve the control problem stated in §II. We use $\mathcal{C}_1$ from Assumption 1 as the first-level controller. The second-level controller, $\mathcal{C}_2$, will determine the reference signal $\check{x}(t)$ to $\mathcal{C}_1$. The controller $\mathcal{C}_2$ functions in discrete time and is described below as Algorithm 1, where $escape$ is a boolean variable. The sampling period by which $\mathcal{C}_2$ operates is $T_s$, and for every continuous signal we use the notation $\cdot_k$ to denotes its sampling at time $kT_s$, as in $x_k \doteq x(kT_s)$. We also note that the reference signal to $\mathcal{C}_1$, $\check{x}$, is constant during each time-interval of length $T_s$:

$$\check{x}(t) = \check{x}_k, \qquad \forall k, t : kT_s \leq t < (k+1)\,T_s. \tag{8}$$

*Theorem 1:* If Assumptions 1, 2 and 3 hold, $(H_0, d_0) \in \mathcal{P}_0$, and $\lim_{t\to\infty} r(t) = r_\infty$ for some fixed $r_\infty \in \mathcal{R}$, then controllers $\mathcal{C}_1$ and $\mathcal{C}_2$ will ensure all trajectories of (1)–(2) are bounded and satisfy

$$\limsup_{t\to\infty} \|c\,(x(t), y(t)) - r_\infty\| \leq \alpha\,(v_{\max}, w_{\max})$$

where $\alpha$ is a continuous function, that may depend on $r_\infty$, and satisfies $\alpha(0,0) = 0$. In the case that the functions $\gamma$ and $c$ are polynomials of degrees $n_\gamma$ and $n_c$, respectively, and $\mathcal{R}$ is compact, then $\alpha$ above can be written independently of $r_\infty$ as a polynomial of degree $n_\gamma + n_c$.

*Algorithm 1 (definition of $\mathcal{C}_2$):*

**input:** $t = kT_s$, $\underline{x}_k$, $\underline{y}_k$, $r_k$
  **if** $k = 0$ **then**
    arbitrarily choose $(\hat{H}_0, \hat{d}_0) \in \mathcal{P}_3$
    set $escape =$ **false**
  **end if**
  **if** $\|\underline{y}_k - (\hat{H}_k \underline{x}_k + \hat{d}_k)\| \leq \delta$ **then**
    set $\hat{H}_{k+1} = \hat{H}_k$, $\hat{d}_{k+1} = \hat{d}_k$
  **else** set

$$(\hat{H}_{k+1}, \hat{d}_{k+1}) = \arg\min_{(H,d)\in\mathcal{P}_3} \|H - \hat{H}_k\|_F^2 + \|d - \hat{d}_k\|^2 \quad \text{s.t. } \|\underline{y}_k - (H\underline{x}_k + d)\| \leq \delta \tag{9}$$

  **end if**
  **if** $(\hat{H}_{k+1}, \hat{d}_{k+1}) \in \mathcal{P}_1$ **then** set $escape =$ **false**
  **else if** $(\hat{H}_{k+1}, \hat{d}_{k+1}) \notin \mathcal{P}_2(r_k)$ **then** $escape =$ **true**
  **end if**
  **if** escape **then** set $\check{x}_{k+1} = x_\nu$
  **else** set $\check{x}_{k+1} = g\,(H_{k+1}, d_{k+1}, r_k)$
  **end if**
**output:** $\check{x}_{k+1}$

## IV. Proof

We define vec : $\mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ as the linear operation converting a matrix to a vector by stacking its rows: $\operatorname{vec}(M) \doteq [M_{11}, \ldots, M_{1n}, \ldots, M_{n1}, \ldots, M_{nn}]^T$. We define

$$a_0 \doteq \begin{pmatrix} \operatorname{vec}(H_0) \\ d_0 \end{pmatrix}, \quad \hat{a}_k \doteq \begin{pmatrix} \operatorname{vec}(\hat{H}_k) \\ \hat{d}_k \end{pmatrix}, \quad \forall k \in \mathbb{N}$$

and say that $\hat{a}_k \in \mathcal{P}_3$ if and only if $(\hat{H}_k, \hat{d}_k) \in \mathcal{P}_3$. With (5) we can write $(H_k \underline{x}_k + d_k) = \underline{X}_k a_k$.

*Lemma 1:* The limit $\lim_{k\to\infty} \hat{a}_k = \hat{a}_\infty \in \mathcal{P}_3$ exists.

*Proof:* Define the nonnegative Lyapunov function,

$$V_k = \|\hat{a}_k - a_0\|^2 . \tag{10}$$

As we show next it is non-increasing with time. By the algorithm if $\|\underline{y}_k - \underline{X}_k a_k\| \leq \delta$ then $a_{k+1} = a_k$ and $V_k$ is non-increasing. Otherwise, note that (9) is a convex optimization problem. Define the convex set $\mathcal{Q}_k \doteq \left\{a \,\middle|\, \|\underline{y}_k - \underline{X}_k a\| \leq \delta\right\}$. In (9) we look for the point that is closest to $a_k$ in the convex intersection of the convex sets $\mathcal{P}_3$ and $\mathcal{Q}_k$. Since $a_k \in \mathcal{P}_3 \setminus \mathcal{Q}_k$ the closest point must lie on $\partial\mathcal{Q}_k$, the boundary of $\mathcal{Q}_k$. Using Assumption 2d we can rewrite (9) as

$$\begin{aligned} \hat{a}_{k+1} = \arg\min_a \quad & f_0(a) \\ \text{s.t.} \quad & f_i(a) \leq 0, \quad i = 1, \ldots, m \\ & f_{m+1}(a) \leq 0, \quad h(a) = 0 \end{aligned} \tag{11}$$

where

$$f_0(a) = (a - \hat{a}_k)^T (a - \hat{a}_k),$$
$$f_{m+1}(a) = (\underline{X}_k a - \underline{y}_k)^T (\underline{X}_k a - \underline{y}_k) - \delta^2.$$

For $a_0$ we have $h\,(a_0) = 0$ and $f_i\,(a_0) \leq 0$, $\forall i = 1, \ldots, m$. We also have

$$\begin{aligned} \|\underline{X}_k a_0 - \underline{y}_k\| &= \|H_0(x_k + v_k) + d_0 - (H_0 x_k + d_0 + w_k)\| \\ &= \|H_0 v_k - w_k\| \leq \delta/2 \end{aligned}$$

by definition of $\delta$ in (7) so $f_{m+1}\,(a_0) < 0$. Let $a' = (1-\varepsilon)\,a_0 + \varepsilon a_s$ where $a_s = (H_s, d_s)$ from Assumption 2d. Convexity of the $f_i$'s implies that for any $\varepsilon \in (0,1]$, $f_i\,(a') < 0$ $\forall i = 1, \ldots, m$. Choosing $\varepsilon$ sufficiently small we can also have $f_{m+1}\,(a') < 0$ which then implies that Slater's constraint qualification holds for (11). Therefore strong duality holds by Slater's theorem and the optimal solution to (9), $\hat{a}_{k+1}$, satisfies the KKT conditions [13, §5.5.3]. This means Lagrange multipliers $\lambda_i \geq 0$ and $\nu \in \mathbb{R}^q$ exist such that

$$\nabla f_0(\hat{a}_{k+1}) + \sum_{\substack{i\in\{1,\ldots,m\} \\ f_i(\hat{a}_{k+1})=0}} \lambda_i \nabla f_i(\hat{a}_{k+1}) + \lambda_{m+1} \nabla f_{m+1}(\hat{a}_{k+1}) + \nu^T \nabla h(\hat{a}_{k+1}) = 0. \tag{12}$$

Define $\tilde{a}_k = \hat{a}_k - a_0$. Choose $i \in \{1, \ldots, m\}$ such that $f_i\,(\hat{a}_{k+1}) = 0$. We have that $f_i\,(a_0) \leq 0$. Convexity of $f_i$ then implies that $\nabla f_i\,(\hat{a}_{k+1})\,\tilde{a}_{k+1} \geq 0$. Since $h$ is affine and $h\,(\hat{a}_{k+1}) = h\,(a_0) = 0$, $\nabla h\,(\hat{a}_{k+1})\,\tilde{a}_{k+1} = 0$. Multiplying both sides of (12) by $\tilde{a}_{k+1}$ we then get that

$$(\tilde{a}_{k+1} - \tilde{a}_k)^T \tilde{a}_{k+1} + \lambda_{m+1} (\underline{X}_k \hat{a}_{k+1} - \underline{y}_k)^T \underline{X}_k \tilde{a}_{k+1} \leq 0. \tag{13}$$

Noting that for some $\|\bar{w}_k\| \le \delta/2$, $\hat{a}_{k+1} \in \partial \mathcal{Q}_k$,

$$\begin{aligned}\underline{X}_k \hat{a}_{k+1} - \underline{y}_k &= \underline{X}_k \hat{a}_{k+1} - \left(\underline{X}_k a_0 - H_0 v_k + w_k\right) \\ &= \underline{X}_k \tilde{a}_{k+1} - \bar{w}_k, \\ \|\underline{X}_k \tilde{a}_{k+1}\| &\ge \|\underline{X}_k \hat{a}_{k+1} - \underline{y}_k\| - \|\bar{w}_k\| \ge \delta/2 \ge \|\bar{w}_k\|,\end{aligned}$$

we get from (13):

$$\begin{aligned}&\tilde{a}_{k+1}^T \tilde{a}_{k+1} - \tilde{a}_k^T \tilde{a}_{k+1} \\ &\quad \le \lambda_{m+1}\left(-\tilde{a}_{k+1} \underline{X}_k^T \underline{X}_k \tilde{a}_{k+1} + \bar{w}_k^T \underline{X}_k \tilde{a}_{k+1}\right) \le 0. \quad (14)\end{aligned}$$

This establishes that $\|\tilde{a}_{k+1}\| \le \|\tilde{a}_k\|$ so the Lyapunov function is indeed non-increasing.

As $V$ is nonnegative and non-increasing, its limit exists and

$$\lim_{k\to\infty} V_k - V_{k+1} = 0. \quad (15)$$

From (14) we also get

$$\begin{aligned}0 \le d_k^2 &\doteq \|\hat{a}_{k+1} - \hat{a}_k\|^2 = \|\tilde{a}_{k+1} - \tilde{a}_k\|^2 \\ &= \|\tilde{a}_{k+1}\|^2 - 2\tilde{a}_k^T \tilde{a}_{k+1} + \|\tilde{a}_k\|^2 \\ &\le \|\tilde{a}_{k+1}\|^2 - 2\|\tilde{a}_{k+1}\|^2 + \|\tilde{a}_k\|^2 = V_k - V_{k+1}. \quad (16)\end{aligned}$$

The statement of the Lemma then follows from (16), (15), and the completeness of $\mathcal{P}_3$. ■

*Lemma 2:* All trajectories of (1)–(2) are bounded and $\limsup_{k\to\infty} \|\underline{X}_k \tilde{a}_\infty\| \le \delta$.

*Proof:* As $g$ is continuous, and $(\hat{H}_k, \hat{d}_k)$ is confined to the compact set $\mathcal{P}_3$, the trajectory of $\check{x}_k \in \{g(\hat{H}_k, \hat{d}_k, r_{k-1}), x_\nu\}$ is bounded. By Assumption 1 this implies all trajectories of (1)–(2) are bounded. Therefore

$$\begin{aligned}\limsup_{k\to\infty} \|\underline{X}_k \tilde{a}_\infty\| \le& \limsup_{k\to\infty} \|\underline{X}_k \tilde{a}_{k+1}\| \\ &+ \limsup_{k\to\infty} \|\underline{X}_k \left(\tilde{a}_\infty - \tilde{a}_{k+1}\right)\| \\ \le& \delta + \limsup_{k\to\infty} \|\underline{X}_k\| \lim_{k\to\infty} \|\tilde{a}_\infty - \tilde{a}_{k+1}\| = \delta.\end{aligned}$$

■

*Lemma 3:* Under the assumptions of Theorem 1, for any $k$ if $(\hat{H}_k, \hat{d}_k) \notin \mathcal{P}_2(r_{k-1})$, then there exists $k' > k$, $k' < \infty$ such that $(\hat{H}_k, \hat{d}_k) \in \mathcal{P}_1$.

*Proof:* Assume the statement of the lemma is not true. Then $\check{x}_{k'} = x_\nu \ \forall k' \ge k$, which implies using (3) that $\limsup_{t\to\infty} \|x(t) - x_\nu\| \le \gamma\left(v_{\max}\right)$. With this, and using Lemma 2,

$$\begin{aligned}\|X_\nu \tilde{a}_\infty\| &= \|(\hat{H}_\infty - H_0)x_\nu + \hat{d}_\infty - d_0\| \\ &= \|(\hat{H}_\infty - H_0)(x_k + v_k) + \hat{d}_\infty \\ &\quad - d_0 + (\hat{H}_\infty - H_0)(x_\nu - (x_k + v_k))\| \\ &\le \limsup \|\underline{X}_k \tilde{a}_\infty\| \\ &\quad + 2\sup_{H\in\mathcal{P}_3} \|H\|(\gamma(v_{\max}) + v_{\max}) \\ &\le \delta + 2\sup_{H\in\mathcal{P}_3} \|H\|(\gamma(v_{\max}) + v_{\max}) \doteq \epsilon'_\nu.\end{aligned}$$

Choose $a$ and $w$ that solves the following set of linear equations

$$\begin{bmatrix} X_\nu & 0 \\ I & X_\nu^T \end{bmatrix} \begin{pmatrix} a \\ w \end{pmatrix} = \begin{pmatrix} X_\nu a_0 \\ \hat{a}_\infty \end{pmatrix}. \quad (17)$$

Note that the matrix in (17) is square and it is easy to see that its null space is zero given that $X_\nu$ has full row rank. Therefore there are $a$ and $w$ satisfying (17). By Assumption 3a, $X_\nu a = X_\nu a_0$ implies $a \in \mathcal{P}_0$. Moreover $\|X_\nu\left(\hat{a}_\infty - a\right)\| = \|X_\nu\left(\hat{a}_\infty - a_0\right)\| \le \epsilon'_\nu$. That and the fact that $\hat{a}_\infty - a = X_\nu^T w$ for some $w$ implies that

$$\|\hat{a}_\infty - a\| \le \frac{1}{\sigma_{\min,>0}\left(X_\nu\right)} \epsilon'_\nu = \epsilon_\nu.$$

Since $a \in \mathcal{P}_0$, $\hat{a}_k \in \mathcal{P}_3 \ \forall k$, and $\hat{a}_k \to \hat{a}_\infty$, by the way $\mathcal{P}_1$ is constructed in Assumption 3b this implies that there must be $k' > k$, $k < \infty$, such that $\hat{a}_{k'} \in \mathcal{P}_1$. This contradicts the assumption that the statement of Lemma 3 is not true. ■

*Lemma 4:* There exists $k_N$ such that $\hat{a}_{k_N} \in \mathcal{P}_1$ and $\forall k > k_N$, $\hat{a}_k \in \mathcal{P}_2\left(r_{k-1}\right)$.

*Proof:* Assume the statement of the lemma is not true, then there exists a subsequence, $a_{n_k}$, such that $\hat{a}_{n_k} \notin \mathcal{P}_2\left(r_{n_k-1}\right) \ \forall n_k$. By Lemma 3, after every $n_k$ there exists $k'$ such that $\hat{a}_{k'} \in \mathcal{P}_1$. Let $n'_k$ be the sequence of these $k'$'s. Since $a_{n_k}$, being a subsequence of $a_k$, converges to $\hat{a}_\infty$, $\hat{a}_\infty \in \left(\mathcal{P}_3 \setminus B_{\varepsilon''}\left(\mathcal{P}_1\right)\right)^c$ where $\mathcal{P}^c$ is the notation for the closure of the set $\mathcal{P}$ and $\varepsilon''$ comes from Assumption 3b. The subsequence $a_{n'_k}$ also converges to $\hat{a}_\infty$. However, since $\mathcal{P}_1$ is closed, this would imply $\hat{a}_\infty \in \mathcal{P}_1$. This is not possible since $\mathcal{P}_1 \cap \left(\mathcal{P}_3 \setminus B_{\varepsilon''}\left(\mathcal{P}_1\right)\right)^c = \emptyset$. Therefore the statement in the Lemma must be true. ■

*Proof of Theorem 1:* The boundedness of the trajectories is established in Lemma 2. Lemma 1 states that $\lim_{k\to\infty} \hat{a}_k = \hat{a}_\infty$. With Assumption 2c, Lemma 4 implies that $\hat{a}_\infty \in \mathcal{P}_2\left(r_\infty\right)$. For every $k > k_N$, where $k_N$ is given by Lemma 4, $\check{x}_k = g(\hat{H}_k, \hat{d}_k, r_{k-1})$. And since $g$ is continuous,

$$\lim_{t\to\infty} \check{x}(t) = \lim_{k\to\infty} \check{x}_k = \check{x}_\infty \doteq g(\hat{H}_\infty, \hat{d}_\infty, r_\infty).$$

The controller $\mathcal{C}_1$ is such that (3) holds. Therefore

$$\limsup_{k\to\infty} \|x_k - \check{x}_\infty\| \le \limsup_{t\to\infty} \|x(t) - \check{x}_\infty\| \le \gamma\left(v_{\max}\right). \quad (18)$$

Because $c$ is continuous, for every $x$, $H$ and $d$, there exist two functions of class $\mathcal{K}$ with respect to their first argument (when all other arguments remain constant), $\gamma_c\left(\varepsilon; x, H, d\right)$ and $\gamma'_c\left(\varepsilon; x, H, d\right)$, such that

$$\begin{aligned}&\|c\left(x, Hx + d + \delta_y\right) - c\left(x, Hx + d\right)\| \\ &\quad \le \gamma_c\left(\|\delta_y\|; x, H, d\right) \qquad \forall \delta_y \in \mathbb{R}^m \\ &\|c\left(x + \delta_x, H\left(x + \delta_x\right) + d\right) - c\left(x, Hx + d\right)\| \\ &\quad \le \gamma'_c\left(\|\varepsilon\|; x, H, d\right) \qquad \forall \delta_x \in \mathbb{R}^n.\end{aligned}$$

Since $\mathcal{P}_0$ and $\mathcal{P}_2\left(r_\infty\right) \cap \mathcal{P}_3$ are compact, the following functions are also of class $\mathcal{K}$:

$$\gamma_c\left(\varepsilon\right) \doteq \sup_{\substack{(H,d)\in\mathcal{P}_2(r_\infty)\cap\mathcal{P}_3 \\ (H',d')\in\mathcal{P}_2(r_\infty)\cap\mathcal{P}_3}} \gamma_c\left(\varepsilon; g\left(H', d', r_\infty\right), H, d\right) \qquad \forall \varepsilon \ge 0$$

$$\gamma'_c\left(\varepsilon\right) \doteq \sup_{\substack{(H,d)\in\mathcal{P}_0 \\ (H',d')\in\mathcal{P}_2(r_\infty)\cap\mathcal{P}_3}} \gamma'_c\left(\varepsilon; g\left(H', d', r_\infty\right), H, d\right) \qquad \forall \varepsilon \ge 0.$$

With that we can write

$$
\begin{aligned}
&\limsup_{t\to\infty} \|c\big(x(t), y(t)\big) - r_\infty\| \\
&\quad = \limsup_{t\to\infty} \|c\big(\check{x}_\infty + (x(t) - \check{x}_\infty)\,, H_0\check{x}_\infty + d_0 \\
&\qquad + H_0\,(x(t) - \check{x}_\infty)\big) - r_\infty\| \\
&\quad \le \gamma_c'\left(\gamma\left(v_{\max}\right)\right) + \|c\left(\check{x}_\infty, H_0\check{x}_\infty + d_0\right) - r_\infty\| \\
&\quad = \gamma_c'\left(\gamma\left(v_{\max}\right)\right) + \|c\big(\check{x}_\infty, H_\infty\check{x}_\infty + d_\infty \\
&\qquad - \left(\left(H_\infty - H_0\right)\check{x}_\infty + d_\infty - d_0\right)\big) - r_\infty\| \\
&\quad \le \gamma_c'\left(\gamma\left(v_{\max}\right)\right) + \gamma_c\big(\|\left(H_\infty - H_0\right)\left(x_k + v_k\right) + d_\infty \\
&\qquad - d_0 - \left(H_\infty - H_0\right)\left(x_k - \check{x}_\infty + v_k\right)\|\big)\ \forall k \\
&\quad \le \gamma_c'\left(\gamma\left(v_{\max}\right)\right) + \gamma_c\Big(\limsup_{k\to\infty}\|\underline{X}_k\tilde{a}_\infty\| \\
&\qquad + \sup_{\substack{H_\infty\in\mathcal{P}_2(r_\infty)\cap\mathcal{P}_3 \\ H_0\in\mathcal{P}_0}} \|H_\infty - H_0\| \times \\
&\qquad \big(\limsup_{k\to\infty}\|x_k - \check{x}_\infty\| + \limsup_{k\to\infty}\|v_k\|\big)\Big) \\
&\quad \le \alpha\left(v_{\max}, w_{\max}\right) \\
&\quad \doteq \gamma_c'(\gamma(v_{\max})) + \gamma_c\Big(\delta\left(v_{\max}, w_{\max}\right) \\
&\qquad + 2\sup_{H\in\mathcal{P}_2(r_\infty)\cap\mathcal{P}_3} \|H\|\,\big(\gamma(v_{\max}) + v_{\max}\big)\Big). \qquad (19)
\end{aligned}
$$

For the last inequality in (19) we used Lemma 2. Since $\delta(\cdot,\cdot)$ is a linear function and $\delta(0,0) = 0$, it is easy to see that $\alpha\left(v_{\max}, w_{\max}\right)$ goes to zero continuously as both $v_{\max}$ and $w_{\max}$ go to zero.

If $c$ is a polynomial of degree $n_c$, then so are $\gamma_c\left(\cdot; x, H, d\right)$ and $\gamma_c'\left(\cdot; x, H, d\right)$ for every $x$, $H$ and $d$. And if further $\mathcal{R}$ is compact, then we can replace $\gamma_c(\cdot)$ and $\gamma_c'(\cdot)$ with their supremum over $\mathcal{R}$, making them independent of $r_\infty$ and also polynomials of degree $n_c$. Finally, if $\gamma$ is a polynomial of degree $n_\gamma$, then $\alpha$, as a composition of two polynomial of degrees $n_c$ and $n_\gamma$, is a polynomial of degree $n_\gamma + n_c$. ■

## V. Application

We now present a practical problem where the objective function is quadratic, and we show that all the assumptions are satisfied. The system we consider is given by (1)-(2) with the following settings: $n = 2$; $H_0$ has the special form

$$H_0 = \begin{bmatrix} Z_R & -Z_X \\ Z_X & Z_R \end{bmatrix},\ Z_R, Z_X \in \mathbb{R}, \qquad (20)$$

with $Z_R \in [0, R_{\max}]$, $Z_X \in [0, \rho_{\max} Z_R]$ and $\sigma \le \|d_0\| \le \varsigma$, where $R_{\max} > 0$, $\rho_{\max} \ge 0$, $\sigma > 0$, $\varsigma > \sigma$ are known; and the control objective is defined by the function $c$

$$c(x, y) \doteq \begin{pmatrix} y_1 x_1 + y_2 x_2 \\ y_2 x_1 - y_1 x_2 \end{pmatrix} \qquad (21)$$

where here $x_i$ denotes the $i$'th element of the vector $x$.

The settings above are typical in induction generator control, where $x$ is the current in a d-q reference frame [14] that exits the generator toward an infinite bus, $y$ is the voltage applied on the generator, $H_0$ represents the unknown impedance of the line connecting the generator to the infinite bus, $d_0$ is the infinite bus voltage, and the function $c$ computes the power being generated. The state $\xi$ represents the rotor current as well as additional state variables such as those of the AC-DC-AC converter in a doubly-fed-induction generator for example, [15], [16]. In this example, the control input $u$ stands for the rotor and the converter voltages.

In strong-grid conditions, where $H_0$ is negligible and the power becomes a linear function of the state $x$, a PI controller taking as input the error between desired and actual power, has been shown to track the reference power, [15]. The same PI controller, with the error between the actual state and the desired state from controller $\mathcal{C}_2$ as the input, can be used as the $\mathcal{C}_1$ controller, satisfying Assumption 1. We assume the external reference signal is attainable, belonging to some properly defined $\mathcal{R}$. In a wind-turbine with an induction-generator application, this means that while the desired active power follows the wind-speed, the reactive power to be generated is sufficiently conservative to support the transfer of the desired active power over all possible line impedances.

For ease of presentation, we define $z \doteq Z_R + jZ_X \in \mathbb{C}$ and $v \doteq d_1 + jd_2 \in \mathbb{C}$. From the problem statement,

$$\mathcal{P}_0 \doteq \left\{ (z, v) \in \mathbb{C}^2 \;\middle|\; \begin{array}{c} 0 \le \operatorname{Re} z \le R_{\max}, \\ 0 \le \operatorname{Im} z \le \rho_{\max} \operatorname{Re} z, \\ \sigma \le |v| \le \varsigma \end{array} \right\}.$$

The set $\mathcal{P}_3$, the convex hull of $\mathcal{P}_0$, is then

$$\mathcal{P}_3 \doteq \left\{ (z, v) \in \mathbb{C}^2 \;\middle|\; \begin{array}{c} 0 \le \operatorname{Re} z \le R_{\max}, \\ 0 \le \operatorname{Im} z \le \rho_{\max} \operatorname{Re} z, \\ |v| \le \varsigma \end{array} \right\}.$$

These sets satisfy Assumptions 2b and 2d.

Defining $\chi \doteq x_1 + jx_2 \in \mathbb{C}$ and $\varrho \doteq r_1 + jr_2 \in \mathbb{C}$, and using $\bar{\cdot}$ for the complex conjugate operation, we note that $c\left(x, Hx + d\right) = r$ can be written as $\left(z\chi + v\right)\bar{\chi} = \varrho$. A solution, in $\chi$, to the latter exists if and only if

$$Q\left(z, v, \varrho\right) \doteq |v|^4 + 4\,|v|^2 \operatorname{Re}\left(\bar{z}\varrho\right) - 4\left(\operatorname{Im}\left(\bar{z}\varrho\right)\right)^2 \ge 0.$$

In this case, when $Q$ is nonnegative, one solution is given by

$$\chi = g\left(z, v, \varrho\right) \doteq -\frac{|v|^2 + 2j\operatorname{Im}\left(\bar{z}\varrho\right) - \sqrt{Q\left(z, v, \varrho\right)}}{2z\bar{v}}. \qquad (22)$$

To make $g$ continuous at $z = 0$, we set $g\left(0, v, \varrho\right) \doteq \bar{\varrho}/\bar{v}$, which is a solution to the control objective if $Q\left(0, v, \varrho\right) \ge 0$. When $\operatorname{Im}\left(\bar{z}\varrho\right) = 0$ there is a solution to the control objective with $v = 0$ since $Q\left(z, 0, \varrho\right) = 0$. However, $g$ can not be made continuous at this point. We therefore define the set $\mathcal{P}_2(r)$ as

$$\mathcal{P}_2(r) = \mathcal{P}_2(\varrho) \doteq \left\{(z, v) \in \mathbb{C}^2 \,|Q\left(z, v, \varrho\right) \ge 0,\, \sigma' \le |v|\right\}$$

where $0 < \sigma' < \sigma$, for which Assumptions 2a and 2c are satisfied. Note that because no point in $\mathcal{P}_0$ violates the constraint $\sigma' \le |v|$, and the definition of $\mathcal{R}$ guarantees that $Q\left(z, v, \varrho\right) \ge 0$ for all $r \in \mathcal{R}$ and for every $(z, v) \in \mathcal{P}_0$, we have that $\mathcal{P}_0 \subset \mathcal{P}_2\left(r\right)$ according to the definition of $\mathcal{P}_2$.

Because $\mathcal{P}_3 \not\subset \mathcal{P}_2\left(r\right)$, we need to also satisfy Assumption 3. For that end we define $x_\nu \doteq [0, 0]^T$. With this definition, (6) is satisfied only if $v = v_0$. Since $\left(z, v_0\right) \in \mathcal{P}_0\ \forall\left(z, v\right) \in \mathcal{P}_3$, Assumption 3a is satisfied. To calculate $\delta$ and $\epsilon_\nu$, we note that

$$\sup_{(H,d)\in\mathcal{P}_2} \|H\| = \sup_{(H,d)\in\mathcal{P}_0} \|H\| = \sup_{\substack{0\le\operatorname{Re} z\le R_{\max} \\ 0\le\operatorname{Im} z\le\rho_{\max}\operatorname{Re} z}} |z| = R_{\max}\sqrt{1 + \rho_{\max}^2},$$

so $\delta = 2(w_{\max} + R_{\max}\sqrt{1+\rho_{\max}^2}v_{\max})$. We also note that with our choice of $\nu$, $\sigma_{\min,>0}(X_\nu) = 1$, so $\epsilon_\nu = 2(w_{\max} + 2R_{\max}\sqrt{1+\rho_{\max}^2}v_{\max})$. Assuming $\epsilon_\nu < \sigma$ define now $\sigma_\epsilon = \sigma - \epsilon_\nu$. Since both the real and imaginary parts of $z$ are nonnegative in $\mathcal{P}_3$, and in the problem statement we assumed the same is true for $\varrho$, we have that $\operatorname{Re}(\bar{z}\varrho) \geq 0$. Therefore if $(z, v) \in \mathcal{P}_0$ then

$$Q(z, v, \varrho) \geq Q_{\sigma_\epsilon}(z, \varrho) \doteq \sigma_\epsilon^4 + 4\sigma_\epsilon^2 \operatorname{Re}(\bar{z}\varrho) - 4(\operatorname{Im}(\bar{z}\varrho))^2 .$$

$Q_{\sigma_\epsilon}(z, \varrho)$ is a quadratic function of $z$ with a negative definite second derivative. This function can be written as a real quadratic function over $\mathbb{R}^2$. Evaluating such a real quadratic function over a convex polygon in $\mathbb{R}^2$, the function will attain its minimum at one of the vertices of the polygon. Projecting $\mathcal{P}_0$ on the space corresponding to $z$, the vertices are $z_1 = 0$, $z_2 = R_{\max}$ and $z_3 = R_{\max} + j\rho_{\max}R_{\max}$. $Q_{\sigma_\epsilon}(z_1, \varrho)$ is always positive. $Q_{\sigma_\epsilon}(z_2, \varrho)$ is nonnegative if

$$0 \leq R_{\max} \leq \frac{{\sigma_\epsilon}^2}{2}\left(\frac{|\varrho| + \operatorname{Re}\varrho}{(\operatorname{Im}\varrho)^2}\right). \tag{23}$$

$Q_{\sigma_\epsilon}(z_3, \varrho)$ is nonnegative if (23) holds and

$$\begin{aligned} 0 \leq& \rho_{\max} \\ \leq& \Big({\sigma_\epsilon}^2 \operatorname{Im}\varrho + 2R_{\max}\operatorname{Re}\varrho \operatorname{Im}\varrho \\ &+ |\varrho|\sqrt{{\sigma_\epsilon}^4 + 4{\sigma_\epsilon}^2 R_{\max}\operatorname{Re}\varrho}\Big) \Big/ \left(2R_{\max}(\operatorname{Re}\varrho)^2\right) . \end{aligned} \tag{24}$$

If (23) and (24) each holds $\forall r \in \mathcal{R}$ with $\sigma_\epsilon$ replaced by $\sigma_\epsilon - \varepsilon$ for some $\varepsilon > 0$, then Assumptions 3b is satisfied. With that we can finally define

$$\mathcal{P}_1 \doteq \left\{ (z, v) \in \mathbb{C}^2 \middle| \begin{array}{c} 0 \leq \operatorname{Re} z \leq R_{\max}, \\ 0 \leq \operatorname{Im} z \leq \rho_{\max}\operatorname{Re} z, \\ \sigma - \epsilon_\nu - \varepsilon/2 \leq |v| \leq \varsigma \end{array} \right\}.$$

*Corollary 1:* If (23) and (24) each holds with $\sigma_\epsilon$ replaced by $\sigma_\epsilon - \varepsilon$ for some $\varepsilon > 0$, then using a PI controller as $\mathcal{C}_1$, and Algorithm 1 with the above definitions of $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $g$ and $x_\nu$ as $\mathcal{C}_2$, guarantees a bounded tracking error as the external reference signal converges. As the bound on the measurement error approaches zero, the tracking error can be made to approach zero as well.

## VI. Conclusions

We considered a control problem which requires that a nonlinear function of the state, with unknown parameters, tracks a given reference signal. We propose a two-level control approach, where a higher level controller estimates the unknown parameters using a discrete adaptive law, and computes the desired steady-state, while a lower level controller drives the plant to this desired steady-state. We confine the parameter search space to a convex set, and use a special escape signal when there is no solution to the tracking problem given the current estimated parameters. Using a dead zone we were also able to prove robustness to measurement errors.

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